The 3D incompressible Euler with a passive scalar: a road to blow-up?

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The 3D incompressible Euler equations with a passive scalar θ are considered in a smooth domain $\Omega \subset \mathbb{R}^3$ with no-normal flow boundary conditions $\boldsymbol{u}\cdot\hat{\boldsymbol{n}}|_{\partial\Omega} = 0$. It is shown that smooth solutions blow up in a finite time if a null (zero) point develops in the vector $\boldsymbol{B} = \nabla q \times \nabla\theta$, provided $\boldsymbol{B}$ has no null points initially: $\boldsymbol{\omega} = \mathrm{curl}\,\boldsymbol{u}$ is the vorticity and $q = \boldsymbol{\omega}\cdot\nabla\theta$ is a potential vorticity. The presence of the passive scalar concentration θ is an essential component of this criterion in detecting the formation of a singularity.

I. INTRODUCTION

It is known that the 3D incompressible Euler equations have an array of very weak solutions [1–8], but whether a singularity develops from smooth initial conditions in a finite time has been a controversial open problem for a generation [9–16]. Most numerical experiments are performed on periodic boundary conditions: the review in [17] cites more than twenty of these. However, the aim of this paper is to study the blow-up problem in the context of the evolution of divergence-free solutions of the Euler equations $\boldsymbol{u}(\boldsymbol{x},t)$ together with a passive scalar $\theta(\boldsymbol{x},t)$

$$\frac{D\boldsymbol{u}}{Dt} = -\nabla p\,, \qquad \frac{D\theta}{Dt} = 0\,, \qquad \frac{D}{Dt} = \partial_t + \boldsymbol{u}\cdot\nabla\,, \qquad \mathrm{div}\,\boldsymbol{u} = 0\,, \tag{1}$$

in a smooth finite domain $\Omega \subset \mathbb{R}^3$ with no-normal flow boundary conditions $\boldsymbol{u}\cdot\hat{\boldsymbol{n}}|_{\partial\Omega} = 0$. The inclusion of θ, which could represent any passive tracer concentration [18–20], allows the vector $\nabla\theta$ to interact with the fluid vorticity field $\boldsymbol{\omega} = \mathrm{curl}\,\boldsymbol{u}$ which evolves according to

$$\frac{D\boldsymbol{\omega}}{Dt} = \boldsymbol{\omega}\cdot\nabla\boldsymbol{u}\,. \tag{2}$$

Formally, it is easily shown that the equivalent of potential vorticity $q = \boldsymbol{\omega}\cdot\nabla\theta$ is also a passive quantity: see [22] for a more general discussion of potential vorticity in the geophysical fluid dynamics context. To show this we write what has become known as Ertel's Theorem as [21]

$$\frac{Dq}{Dt} = \left(\frac{D\boldsymbol{\omega}}{Dt} - \boldsymbol{\omega}\cdot\nabla\boldsymbol{u}\right)\cdot\nabla\theta + \boldsymbol{\omega}\cdot\nabla\left(\frac{D\theta}{Dt}\right)\,, \tag{3}$$

which is no more than a re-arrangement of terms after an application of the product rule. Clearly

$$\frac{Dq}{Dt} = 0\,. \tag{4}$$

A result of Kurgansky [23–25] (see also [26–28]) can now be invoked for any two passive scalars whose gradients define the vector

$$\boldsymbol{B} = \nabla q \times \nabla\theta\,, \tag{5}$$

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in which case $\boldsymbol{B}$ turns out to satisfy

$$\frac{D\boldsymbol{B}}{Dt} = \boldsymbol{B}\cdot\nabla\boldsymbol{u}\,, \qquad \mathrm{div}\,\boldsymbol{B} = 0\,, \qquad \mathrm{div}\,\boldsymbol{u} = 0\,. \tag{6}$$

The $\boldsymbol{B}$-field is a cross-product of two normals to the material surfaces on which θ and q ride and is thus tangent to the curve formed from the intersection of the two surfaces [26, 27]. This result is formally true for the gradient of any two passive scalars riding on a divergence-free Euler flow and is not dependent upon the definition of q used in this context, although the passivity of q is, of course, dependent on this. The key point here is that $\boldsymbol{B}$ contains the *gradient* of $\boldsymbol{\omega}$ (in projection) and two gradients of θ. We propose to exploit the fact that the evolution of $\boldsymbol{B}$ in (6) takes the same form as that of the Euler vorticity field in (2) or a magnetic field in MHD [29, 30].

II. STATEMENT OF THE RESULT

The Beale-Kato-Majda (BKM) theorem is a key result for the 3D Euler equations [9]. A subsequent modification was proved by Ponce [31] in terms of the rate of strain matrix (deformation tensor) defined by $\mathcal{S} = \frac{1}{2}\left(\nabla\boldsymbol{u} + \nabla\boldsymbol{u}^T\right)$:

Theorem 1 *There exists a global solution of the 3D Euler equations $\boldsymbol{u}\in C([0,\,\infty];H^s)\cap C^1([0,\,\infty];H^{s-1})$ for $s\geq 3$ if, for every $T>0$*

$$\int_0^T \|\mathcal{S}(\tau)\|_{L^\infty(\mathbb{R}^3)}\,d\tau < \infty\,. \tag{7}$$

Conversely, if there exists a time T for which

$$\int_0^T \|\mathcal{S}(\tau)\|_{L^\infty(\mathbb{R}^3)}\,d\tau = \infty\,, \tag{8}$$

then $\lim_{t\to T}\|\mathcal{S}(t)\|_{L^\infty(\mathbb{R}^3)} = \infty$.

In the original BKM-result [9] $\|\boldsymbol{\omega}\|_{L^\infty(\mathbb{R}^3)}$ replaced $\|\mathcal{S}\|_{L^\infty(\mathbb{R}^3)}$. The proofs in [9, 31] are valid for flow in all $\Omega\equiv\mathbb{R}^3$ but the techniques used in those papers, such as Fourier transforms and the Biot-Savart integral, are not readily adaptable to no-normal flow boundary conditions $\boldsymbol{u}\cdot\hat{\boldsymbol{n}}|_{\partial\Omega} = 0$. Ferrari [32] circumvented this difficulty by adapting some ideas from the theory of linear elliptic systems to achieve these results for the boundary conditions $\boldsymbol{u}\cdot\hat{\boldsymbol{n}}|_{\partial\Omega} = 0$. For our purposes, (8) is the key blow-up result for our finite domain Ω.

While θ itself is a constant of the motion its gradient can easily be shown to satisfy (likewise for ∇q)

$$\|\nabla\theta(t)\|_{L^\infty(\Omega)} \leq \|\nabla\theta(0)\|_{L^\infty(\Omega)}\exp\int_0^t \|\mathcal{S}\|_{L^\infty(\Omega)}\,d\tau\,. \tag{9}$$

Hence $\int_0^t \|\mathcal{S}\|_{L^\infty(\Omega)}\,d\tau$ controls these gradients as it does solutions of the Euler equations. The main questions revolve around the occurrence of null points (zeros) in $|\boldsymbol{B}|$. Firstly, initial data for $|\boldsymbol{B}|$ must be free of null points: then a null can potentially develop either by maxima or minima developing in θ or q or if ∇q and $\nabla\theta$ momentarily align at some point. §III contains an example of simple initial data and a domain Ω with no null points for $|\boldsymbol{B}|$. In the following, t^* is designated as the earliest time a null point occurs in $|\boldsymbol{B}|$. It is, of course, possible that $\boldsymbol{B}$ could blow up earlier than t^* by some other mechanism.

Theorem 2 *On a smooth domain $\Omega\subset\mathbb{R}^3$ with boundary conditions $\boldsymbol{u}\cdot\hat{\boldsymbol{n}}|_{\partial\Omega} = 0$, with initial data for which $|\boldsymbol{B}(\boldsymbol{x},\,0)| > 0$ and $\|\boldsymbol{B}\,(\boldsymbol{x}\,,0)\,\|_{L^\infty(\Omega)} < \infty$, if there exists a smooth solution of the 3D Euler equations in the interval $[0,\,t^*)$, then at the earliest time t^* at which $|\boldsymbol{B}(\boldsymbol{x},\,t^*)| = 0$*

$$\int_0^{t^*} \|\mathcal{S}\|_{L^\infty(\Omega)}\,d\tau = \infty\,. \tag{10}$$

Conversely, if $\int_0^T \|\mathcal{S}\|_{L^\infty(\Omega)}\,d\tau < \infty$ in any interval $[0,\,T]$ then $|\boldsymbol{B}(\boldsymbol{x},\,t)|$ cannot develop a null point for $t\in[0,\,T]$.

Proof: On the interval $[0,\,t^*)$ first take the scalar product of (5) with $\boldsymbol{B}$, where $|\boldsymbol{B}| = B$

$$\tfrac{1}{2}\frac{D\left(B^2\right)}{Dt} = \boldsymbol{B}\cdot\nabla\boldsymbol{u}\cdot\boldsymbol{B}\,. \tag{11}$$

Thus, dividing by $B^2 = \boldsymbol{B}\cdot\boldsymbol{B}$ and then multiplying by $\ln B$, which could take either sign, we obtain

$$\tfrac{1}{2}\frac{D|\ln B|^2}{Dt} = (\ln B)\left(\hat{\boldsymbol{B}}\cdot\mathcal{S}\cdot\hat{\boldsymbol{B}}\right). \tag{12}$$

Now multiply by $|\ln B|^{2(m-1)}$ for $1 \le m < \infty$

$$\frac{1}{2m}\frac{D|\ln B|^{2m}}{Dt} = |\ln B|^{2(m-1)}(\ln B)\left(\hat{\boldsymbol{B}}\cdot\mathcal{S}\cdot\hat{\boldsymbol{B}}\right), \tag{13}$$

and then integrate over the volume, invoke the Divergence Theorem and the boundary conditions on Ω and finally use Hölder's inequality to obtain

$$\begin{aligned}\frac{1}{2m}\frac{d}{dt}\int_\Omega |\ln B|^{2m}dV &\le \int_\Omega |\ln B|^{2m-1}|\mathcal{S}|\,dV\\ &\le \left(\int_\Omega |\ln B|^{2m}dV\right)^{\frac{2m-1}{2m}}\left(\int_\Omega |\mathcal{S}|^{2m}dV\right)^{\frac{1}{2m}}.\end{aligned} \tag{14}$$

Using the standard notation $\|X\|_{L^p(\Omega)} = \left(\int_\Omega |X|^p dV\right)^{1/p}$, (14) reduces to

$$\frac{d}{dt}\|\ln B\|_{L^{2m}(\Omega)} \le \|\mathcal{S}\|_{L^{2m}(\Omega)} \tag{15}$$

which integrates to

$$\|\ln B(t)\|_{L^{2m}(\Omega)} \le \|\ln B(0)\|_{L^{2m}(\Omega)} + \int_0^t \|\mathcal{S}(\tau)\|_{L^{2m}(\Omega)}\,d\tau\,. \tag{16}$$

Since Ω is bounded we take the limit $m\to\infty$ to obtain

$$\|\ln B(t)\|_{L^{\infty}(\Omega)} \le \|\ln B(0)\|_{L^{\infty}(\Omega)} + \int_0^t \|\mathcal{S}(\tau)\|_{L^{\infty}(\Omega)}\,d\tau\,. \tag{17}$$

Provided $\boldsymbol{B}$ has no zero in its initial data the log-singularity at $|\boldsymbol{B}| = 0$ causes the left hand side to blow up at t^* thereby forcing $\int_0^{t^*}\|\mathcal{S}\|_{L^\infty(\Omega)}\,d\tau \to \infty$ as $t \to t^*$.

Finally, it is immediately clear from (17) that if $\int_0^T \|\mathcal{S}\|_{L^\infty(\Omega)}\,d\tau$ remains finite in an interval $t\in[0,\,T]$ then no null can develop in $\boldsymbol{B}$. ∎

Remark: The scalar product within q and the subsequent vector product of the two gradients within $\boldsymbol{B}$ produce a rich set of possibilities when zeros form in $\nabla\theta$ and ∇q, while $\|\boldsymbol{\omega}\|_{L^\infty(\Omega)}$ simultaneously blows up. For instance, in the case when $\nabla\theta = 0$, while $\|\boldsymbol{\omega}\|_{L^\infty(\Omega)}$ certainly blows up at t^*, it is not clear whether ∇q blows up or not because of the scalar product within q. However, if it does then this is consistent with the equivalent of (9) for ∇q. In the case when a null forms through a maximum or minimum in q, any simultaneous blow-up in q would obviously have to happen elsewhere in the domain other than the null point. Likewise, inequality (17) is consistent with a blow-up in $\boldsymbol{B}$ which, if it occurred, would again have to occur elsewhere than the null point.

III. AN EXAMPLE OF INITIAL DATA WITH NO NULL POINTS

An important question is whether there is initial data such that $|\boldsymbol{B}(\boldsymbol{x},\,0)| > 0$. We proceed to find a simple example of a set of initial data $\boldsymbol{u}$ on a finite domain $\Omega\subset\mathbb{R}^3$ from initial data on $\boldsymbol{\omega}$ and θ such that $|\boldsymbol{B}| > 0$ and $\|\boldsymbol{B}\|_{L^\infty} < \infty$ for the elliptic system

$$\mathrm{curl}\,\boldsymbol{u} = \boldsymbol{\omega}\,, \qquad \mathrm{div}\,\boldsymbol{u} = 0\,, \qquad \boldsymbol{u}\cdot\hat{\boldsymbol{n}}|_{\partial\Omega} = 0\,. \tag{18}$$

The usual methods, such as the Biot-Savart integral, are hard to apply on this domain but for the elliptic system in (18), Ferrari [32] has shown that for given a vector $\boldsymbol{\omega}$, the velocity field $\boldsymbol{u}$ can, in principle, always be constructed. In the next paragraph this construction is performed in an explicit example in which Ω will be determined later.

Take the example $\boldsymbol{\omega} = (1,\,1,\,1)^T$: we firstly observe that there is a velocity field $\boldsymbol{v} = (z,\,x,\,y)^T$ which satisfies $\operatorname{div}\boldsymbol{v} = 0$ and $\operatorname{curl}\boldsymbol{v} = (1,1,1)^T$ but we cannot be sure that it satisfies $\boldsymbol{v}\cdot\hat{\boldsymbol{n}} = 0$ for any given domain Ω. Therefore it needs to be modified to satisfy the boundary conditions. To do this we introduce some potential ϕ such that

$$\boldsymbol{u} = \boldsymbol{v} + \nabla\phi\,. \tag{19}$$

Note that $\operatorname{curl}\boldsymbol{u} = (1,\,1,\,1)^T$. To guarantee that (18) holds, ϕ must satisfy the Neumann boundary value problem

$$\Delta\phi = 0\,, \qquad\qquad \left.\frac{\partial\phi}{\partial n}\right|_{\partial\Omega} = -(z,\,x,\,y)^T\cdot\hat{\boldsymbol{n}}\,, \tag{20}$$

which always has a solution on any smooth domain Ω. Thus we have been able to construct a velocity field $\boldsymbol{u}$ corresponding to $\boldsymbol{\omega} = (1,\,1,\,1)^T$, that satisfies the boundary conditions. For simplicity, now choose $\theta = \frac{1}{2}(x^2+y^2+z^2)$ (say) and calculate q, ∇q and $\nabla\theta$

$$q = x + y + z\,, \qquad \nabla\theta = (x,\,y\,,z)^T\,, \qquad \nabla q = (1,\,1,\,1)^T\,, \tag{21}$$

and then $\boldsymbol{B}$

$$\boldsymbol{B} = (z-y,\,x-z,\,y-x)^T\,. \tag{22}$$

Note that $|\boldsymbol{B}| = 0$ only on the straight line $x = y = z = t$ for $t\in\mathbb{R}$. Hence $|\boldsymbol{B}| > 0$ on any smooth, finite domain Ω that avoids this line: this is enough to achieve our goal.

IV. CONCLUSION

These results raise curious questions regarding the nature of 3D Euler flow with a passive scalar. Physically θ could represent, for instance, the concentration of a dye or a quantity of fine dust added to an Euler flow. As a passive quantity it would be appear to be innocuous but its presence introduces the gradient $\nabla\theta$ which interacts with $\boldsymbol{\omega}$ and thereby introduces the second passive quantity $q = \boldsymbol{\omega}\cdot\nabla\theta$ into the dynamics. The key result is the stretching relation for $\boldsymbol{B}$ in (6), where $\boldsymbol{B}$ is simply a vector tangent to the curve that intersects the two material surfaces for θ and q. The first null point in $|\boldsymbol{B}|$ then drives $\int_0^t \|\mathcal{S}\|_{L^\infty(\Omega)}\,d\tau \to \infty$ through the logarithmic singularity. The presence of θ is therefore essential to this mechanism. This raises the question whether this singularity is of a fundamentally different type than those that are thought to develop in bare 3D Euler flow with no passive scalar present?

The proof of Theorem 2 shows that it is essential that $|\boldsymbol{B}|$ starts with no null points, which rules out the use of periodic boundary conditions because $\boldsymbol{B} = \nabla q\times\nabla\theta$ has zeros for every value of t. Hence a comparison with the main body of numerical experiments is not possible [12–16], although it would suggest that a numerical examination of the occurrence and nature of null points might be fruitful with the boundary conditions used in this paper.

A further variation on this problem is that of the 3D Euler equations with buoyancy, which can be written in the following dimensionless form

$$\frac{D\boldsymbol{u}}{Dt} + \theta\hat{\boldsymbol{k}} = -\nabla p\,, \qquad\qquad \frac{D\theta}{Dt} = 0\,, \qquad\qquad \operatorname{div}\boldsymbol{u} = 0\,. \tag{23}$$

θ, while still passive, is physically a dimensionless temperature and appears because the density has beenn taken to be proportional to θ in the Boussinesq approximation. This changes equation (2) to

$$\frac{D\boldsymbol{\omega}}{Dt} + \nabla\theta\times\hat{\boldsymbol{k}} = \boldsymbol{\omega}\cdot\nabla\boldsymbol{u}\,. \tag{24}$$

The extra term $\nabla\theta\times\hat{\boldsymbol{k}}$ makes no contribution to equation (4) and so q remains passive. However, the BKM criterion for this system on a finite smooth domain Ω would need re-working because of the presence of the $\hat{\boldsymbol{k}}\theta$ buoyancy term in (23).

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